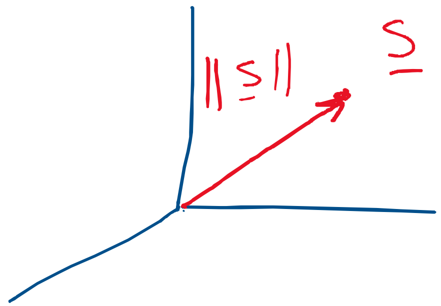
In the Name of GOD

In the previous sesson, we talked about orthonormal expansion of signals with limited energy. we found that Using this expansion, we can represents any signal with limited energy as a point in equivalent vector space.

$$s(t) = \sum_{i=1}^{N} s_i \, \varphi_i(t), \qquad s_i = \langle s(t), \varphi_i(t) \rangle$$

$$s_i = \int_a^b s(t)\, \varphi_i^*(t)\, dt, \qquad \underline{s} = (s_1, s_2, \ldots, s_N)^T$$

$$\|\underline{s}\| = \left(\sum_{i=1}^{N} |s_i|^2 \right)^{1/2} = \sqrt{E_s}$$



N-dimensional vector space

$$E_s = \int_a^b |s(t)|^2 \, dt$$

Gram-Schemith algorithm in signal space

Now, we want to find the Connection between signal space spanned by signal $\{S_m(t)\}_{m=1}^{M}$ and its equivalent vector space. We know that if we can find a set of complete orthonormal basis functions for the signal space spanned by $\{S_m(t)\}_{m=1}^{M}$, the we can represent all the signals in this signal space as the linear combination of orthonormal basis functions $\{\varphi_i(t)\}_{i=1}^{N}$. The Gram-Schemith algorithm help us finding this complete set of orthonormal basis functions $\{\varphi_i(t)\}_{i=1}^{N}$. The Gram-Schemith algorithm works as follows.

Gram. schmith algorithm in signal space, is similar to what we see in the vector space.

1) Select one of the signals in the signal set $\{S_m(t)\}_{m=1}^{M}$ and normalize it and get it as the first basis func.

$$\varphi_1(t) = \frac{S_1(t)}{\underbrace{\|S_1(t)\|}_{\sqrt{E_{S_1}}}} = \frac{S_1(t)}{\sqrt{E_{S_1}}}$$

2) Select another signal from the set $\{S_m(t)\}_{m=1}^{M}$ which is not selected in the previous stages. Then subtract the projection of this selected signal onto previously obtained basis functions, Normalized the result and take it as the next basis function.

for example, if we select $s_2(t)$

$$s_2'(t) = s_2(t) - \underbrace{\langle s_2(t), \varphi_1(t) \rangle}_{c_{12}} \varphi_1(t)$$

$$\varphi_2(t) = \frac{s_2'(t)}{\| s_2'(t) \|} = \frac{s_2'(t)}{\sqrt{E_{s_2'}}}$$

$$\langle s_2(t), \varphi_1(t) \rangle = \int_a^b s_2(t)\, \varphi_1^*(t)\, dt = c_{12}$$

or in the k^{th} level of the algorithm, if we select $s_k(t)$

$$s_k'(t) = s_k(t) - \langle s_k(t), \varphi_1(t) \rangle \varphi_1(t) - \cdots - \langle s_k(t), \varphi_{k-1}(t) \rangle \varphi_k(t)$$

$$s_k'(t) = s_k(t) - \sum_{i=1}^{k-1} \underbrace{\langle s_k(t), \varphi_i(t) \rangle}_{c_{ik}} \varphi_i(t) = s_k(t) - \sum_{i=1}^{k-1} c_{ik}\varphi_i(t)$$

$$\varphi_k(t) = \frac{s_k'(t)}{\| s_k'(t) \|} = \frac{s_k'(t)}{\sqrt{\mathcal{E}_{s_k'}}}$$

Do simplification as an assignment

in which

$$E_{s'_k} = \| s'_k(t) \|^2 = \| s_k(t) - \sum_{i=1}^{k-1} c_{ik} \varphi_i(t) \|^2 = E_{s_k} - \sum_{i=1}^{k-1} |c_{ik}|^2$$

3) Repeat the second stage of the algorithm, till all the signals in the set $\{s_m(t)\}_{m=1}^{M}$ exhausted.

Finally we will find a set of Complete orthonormal basis function $\{\varphi_i(t)\}_{i=1}^{N}$. We know that $N \leq M$

* If in the l th stage of the algorithm, we find that $E'_{s_l} = 0$, it means that $s_l(t)$ is a linear Combination of the other signals in the set $\{S_m(t)\}_{m=1}^{M}$ $\neq S_l$, it does not give any basis function.

* So, if the signal $\{S_m(t)\}_{m=1}^{M}$ are linearly independent the $N = M$

* As you know, N is the dimension of equivalent vector space for the signal space spanned by $\{S_m(t)\}_{m=1}^{M}$

* Using the Complete orthonormal basis function $\{\varphi_i(t)\}_{i=1}^{N}$, we can represent all the signals in the signal space spanned by $\{S_m(t)\}_{m=1}^{M}$ as a linear Combination of $\{\varphi_i(t)\}_{i=1}^{N}$, means all signals in the set $\{S_m(t)\}_{m=1}^{M}$ can also be represented by the linear Combination of $\{\varphi_i(t)\}_{i=1}^{N}$

$$s_m(t) = \sum_{i=1}^{N} s_{m_i} \varphi_i(t) \quad \text{in which}$$

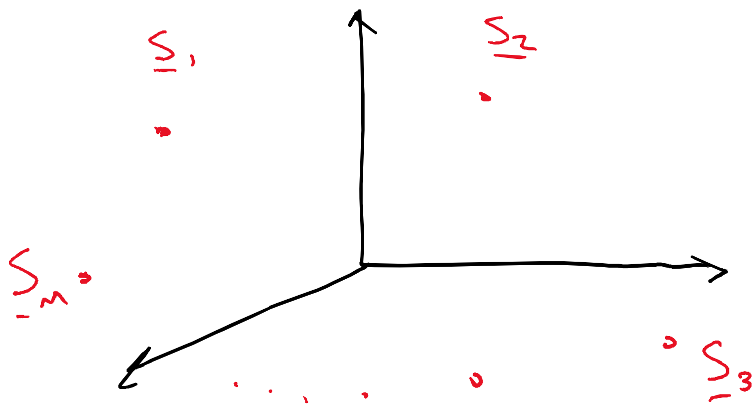
$$s_{m_i} = \langle s_m(t), \varphi_i(t) \rangle = \int_a^b s_m(t)\, \varphi_i^*(t)\, dt$$

$$\xrightarrow[\text{rep. for } s_m(t)]{\text{vector}} \qquad \underline{s}_m = \left(s_{m_1}, s_{m_2}, \ldots, s_{m_N} \right)^T$$

$$E_{s_m} = \int_a^b |s_m(t)|^2\, dt = \| \underline{s}_m \|^2 = \sum_{i=1}^{N} |s_{m_i}|^2$$

So, the signals in the set $\{s_m(t)\}_{m=1}^{M}$ can rep. by M vectors $\{\underline{s}_m\}_{m=1}^{M}$ in the equivalent N-dim vector space

We call it



$\underbrace{\text{Signals}}$ Constellation.

$\{s_m(t)\}_{m=1}^{M}$

Some Comments on Gram-Schemith algorithm

1) When you want to start the algorithm for a set of signals $\{S_m(t)\}_{m=1}^{M}$, you can have a simple look at these signals. If you find that one or more signals in this signal set, are as the linear combination of the other signals in the signal set, you can get these kind of signal out of the

algorithm. Because for these signals $\epsilon_{s'_e} = 0$ and do not give us any basis function.

2) When you want to run the algorithm on the signal set $\{s_m(t)\}_{m=1}^{M}$, you can first have a look at these signals. If you find that two or more signals in this signals set, are orthogonal to each other, you can simply

normalized these signals and take them as some basis functions. This operation will simlify the algorithm.

3) In the Gram-schemith algorithm, based on the order of selected signal in different stages of algorithm, we may find different set of orthonormal basis

functions. It means that the result of Gram-Schmidt algorithm, is not unique. It is important to note that if we found two different orthonormal set as $\left\{\varphi_i(t)\right\}_{i=1}^{N}$ and $\left\{\varphi_i'(t)\right\}_{i=1}^{N}$, Elements of these two sets are linear Combination of each other. Means there is a matrix A, such that

$$[\varphi_1(t), \varphi_2(t), \dots, \varphi_N(t)]^T = A\,[\varphi'_1(t), \varphi'_2(t), \dots, \varphi'_N(t)]^T$$

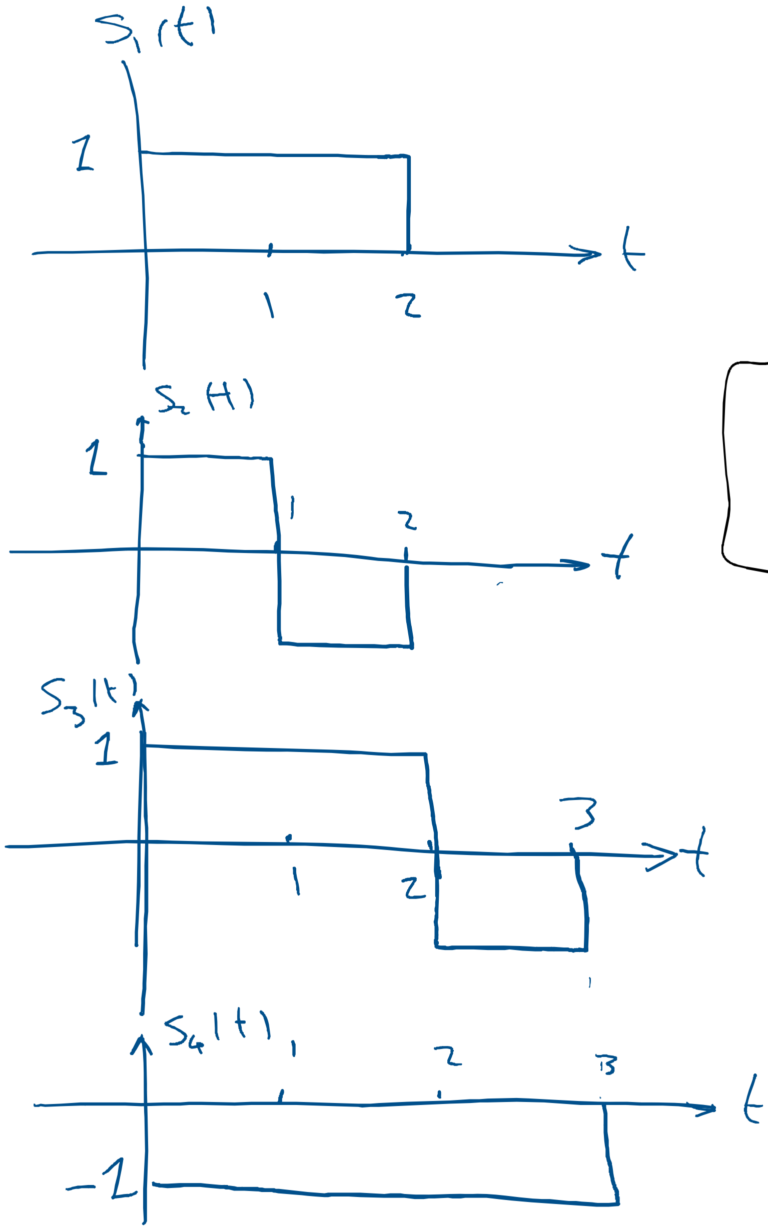
So, their related vector spaces are rotationed (and shifted) version of each other. So, the $\overbrace{\text{dimension}}^{N}$ of these vector space is equal to each other. Besides, vector lengths, distance between signals and inner product of any two signals are fixed in different related vector space.

$$\langle \underbrace{s_m(t), s_\ell(t)}_{\text{Signal space}} \rangle = \langle \underbrace{\underline{s}_m, \underline{s}_\ell}_{\text{Vector space}} \rangle$$

We can see these facts throght an example.

Example :

We want to use Gram-Schmidth algorithm to find a vector rep. of these signals (signal Constellation)



$$\{ s_m(t) \}_{m=1}^{4}$$

$$\boxed{\begin{array}{c} 0 \leq t \leq 3 \\ a \leq t \leq b \end{array}}$$

1) we select $s_1(t)$, normalize it and take it as the firs basis function

$$\varphi_1(t) = \frac{s_1(t)}{\|s_1(t)\|} = \frac{s_1(t)}{\sqrt{E_{s_1}}}$$

$$E_{s_1} = \int_a^b |s_1(t)|^2 dt = \int_0^3 |s_1(t)|^2 dt = \int_0^2 1\, dt = 2$$

$$\Longrightarrow \boxed{\varphi_1(t) = \frac{s_1(t)}{\sqrt{2}}}$$

2) we select $s_2(t)$, subtract the projecton of $s_2(t)$ on $\varphi_1(t)$, normalize the result and take it as the second basis function

$$s_2'(t) = s_2(t) - \langle s_2(t), \varphi_1(t) \rangle \varphi_1(t)$$

$$\langle s_2(t), \varphi_1(t) \rangle = \int_a^b s_2(t)\,\varphi_1(t)\,dt = 0$$

$$\Rightarrow \quad s_2'(t) = s_2(t) \quad \Longrightarrow \quad \varphi_2(t) = \frac{s_2'(t)}{\|s_2'(t)\|} = \frac{s_2(t)}{\|s_2(t)\|} = \frac{s_2(t)}{\sqrt{2}}$$

$$\Longrightarrow \boxed{\varphi_2(t) = \frac{s_2(t)}{\sqrt{2}}}$$

$$\langle s_1(t), s_2(t) \rangle = 0$$

$$s_1(t) \perp s_2(t)$$

3) We use $s_3(t)$, subtract the projection of $s_3(t)$ on $\varphi_1(t)$ & $\varphi_2(t)$, normalize the result & take it as the third basis function.

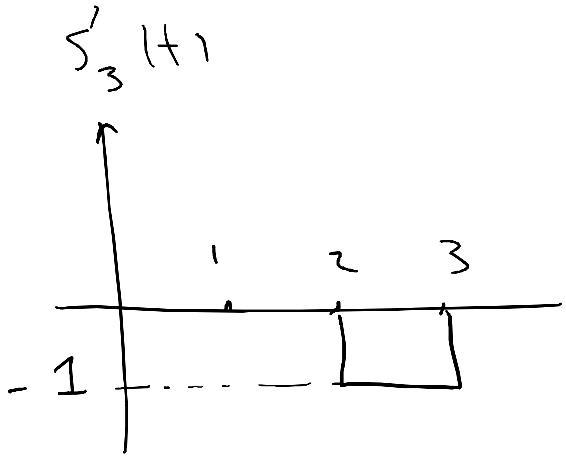
$$s_3'(t) = s_3(t) - \langle s_3(t), \varphi_1(t) \rangle \varphi_1(t) - \langle s_3(t), \varphi_2(t) \rangle \varphi_2(t)$$

$$\langle s_3(t), \varphi_1(t) \rangle = \int_a^b s_3(t)\,\varphi_1(t)\,dt = \int_0^2 \frac{1}{\sqrt{2}}\, 1 \; dt = \sqrt{2}$$

$$\langle s_3(t), \varphi_2(t) \rangle = \int_a^b s_3(t)\,\varphi_2(t)\,dt = \int_0^1 \frac{1}{\sqrt{2}}\, 1 \; dt + \int_1^2 \frac{1}{\sqrt{2}}(-1)\,dt$$

$$= 0$$

$$\Longrightarrow \; s_3'(t) = s_3(t) - \overbrace{\sqrt{2}\,\varphi_1(t)}^{s_1(t)} - 0\,\varphi_2(t) = \begin{cases} 0 & 0 \leq t \leq 2 \\ -1 & 2 \leq t \leq 3 \end{cases}$$



$$\Longrightarrow \quad \varphi_3(t) = \frac{s_3'(t)}{\| s_3'(t) \|} = s_3'(t)$$

$$\| s_3'(t) \| = 1$$

4) we select $s_4(t)$ and subtract the projection of $s_4(t)$ on $\varphi_1(t)$ & $\varphi_2(t)$ & $\varphi_3(t)$, normalize the result and take it as the forth basis function.

$$s_4'(t) = s_4(t) - \langle s_4(t), \varphi_1(t) \rangle \varphi_1(t) - \langle s_4(t), \varphi_2(t) \rangle \varphi_2(t)$$

$$- \langle s_4(t), \varphi_3(t) \rangle \varphi_3(t)$$

$$\langle s_4(t), \varphi_1(t) \rangle = \int_0^2 \frac{1}{\sqrt{2}} (-1)\, dt = -\sqrt{2}$$

$$\langle s_4(t), \varphi_2(t) \rangle = \int_0^1 \frac{1}{\sqrt{2}} (-1)\, dt + \int_1^2 \frac{1}{\sqrt{2}} (+1)\, dt = 0$$

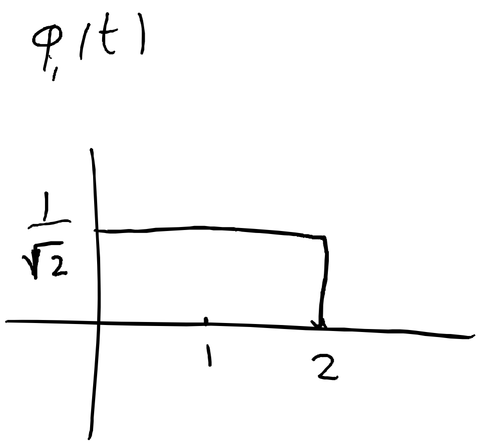
$$\langle s_4(t), \varphi_3(t)\rangle = \int_2^3 1\,dt = 1$$

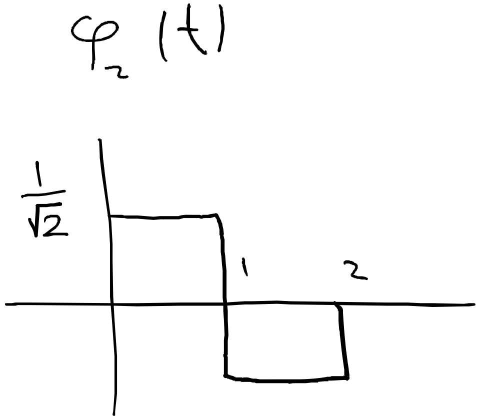
$$\Rightarrow s_4'(t) = s_4(t) - \underbrace{\sqrt{2}\,\varphi_1(t)}_{s_1(t)} - 0\,\varphi_2(t) - \underbrace{\varphi_3(t)}_{s_3'(t)} = 0$$

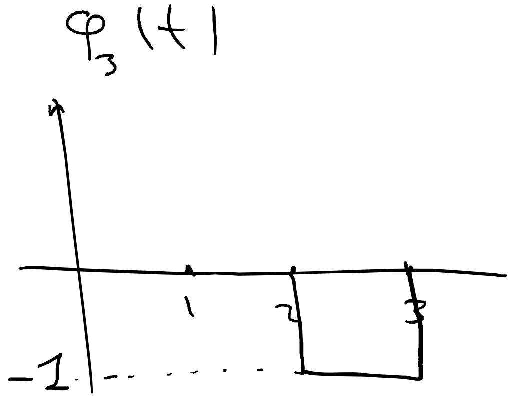
Means that $s_4(t)$ is a linear Combination of

$s_1(t)$, $s_2(t)$ and $s_3(t)$ $\Longrightarrow$ does not give us any basis function.

Finished.







$N = 3$, $M = 4$, $N < M$

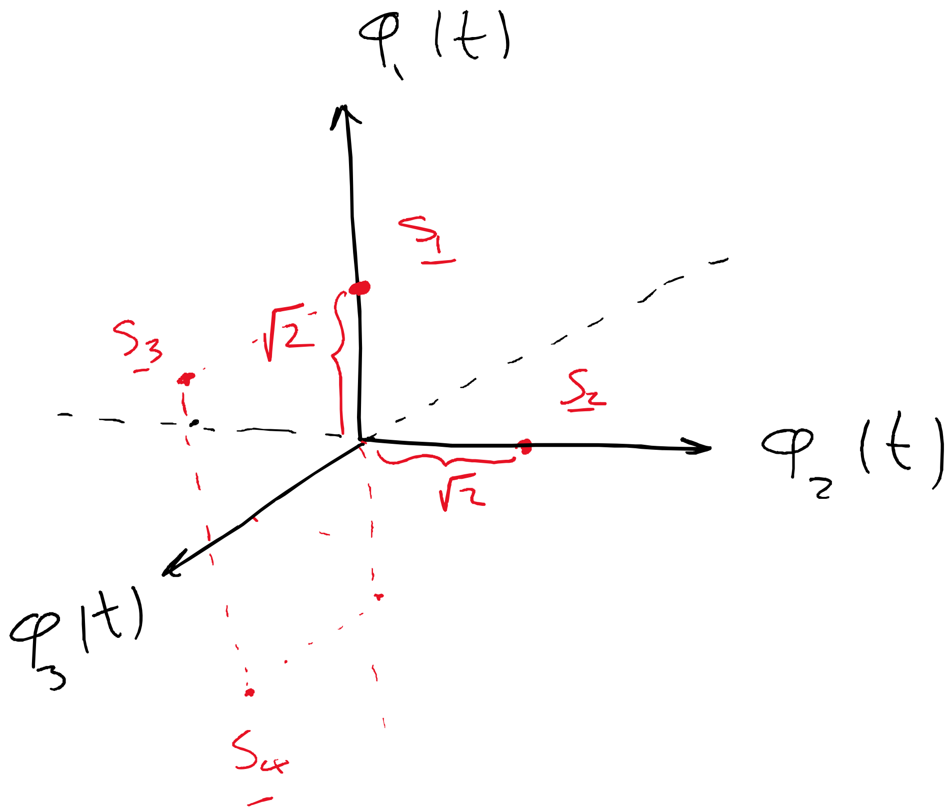
Constellation

$$s_1(t) = \sqrt{2}\,\varphi_1(t) \Longrightarrow \underline{s}_1 = (\sqrt{2}, 0, 0)^T$$

$$s_2(t) = \sqrt{2}\,\varphi_2(t) \Longrightarrow \underline{s}_2 = (0, \sqrt{2}, 0)^T$$

$$s_3(t) = \sqrt{2}\,\varphi_1(t) + 0\,\varphi_2(t) + 1\,\varphi_3(t) \Longrightarrow \underline{s}_3 = (\sqrt{2}, 0, 1)^T$$

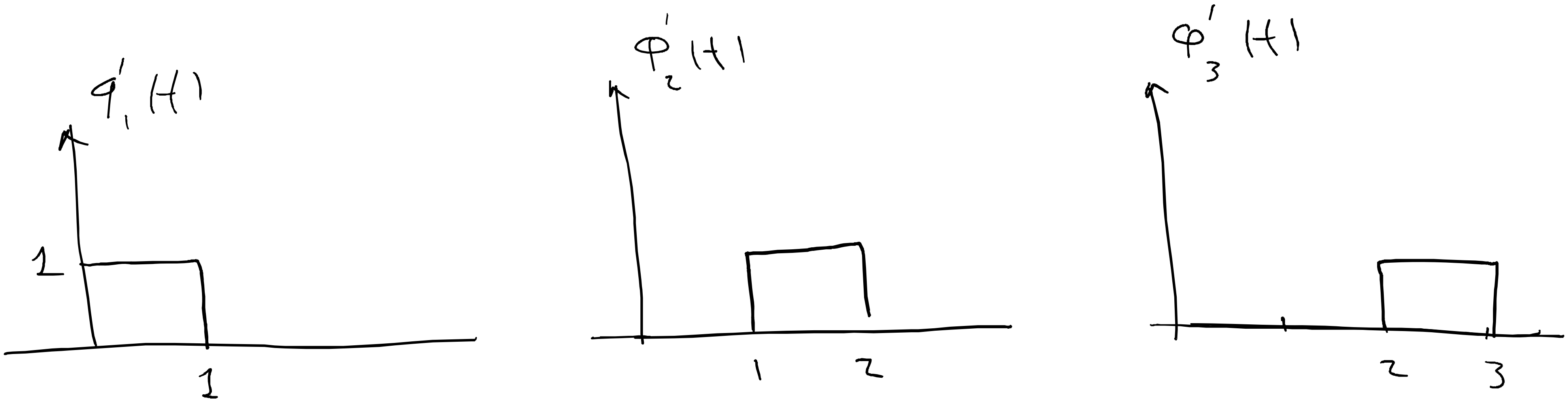
$$s_4(t) = -\sqrt{2}\,\varphi_1(t) + 0\,\varphi_2(t) + 1\,\varphi_3(t) \Longrightarrow \underline{s}_4 = (-\sqrt{2}, 0, 1)^T$$



Sketch

Signal Constellation

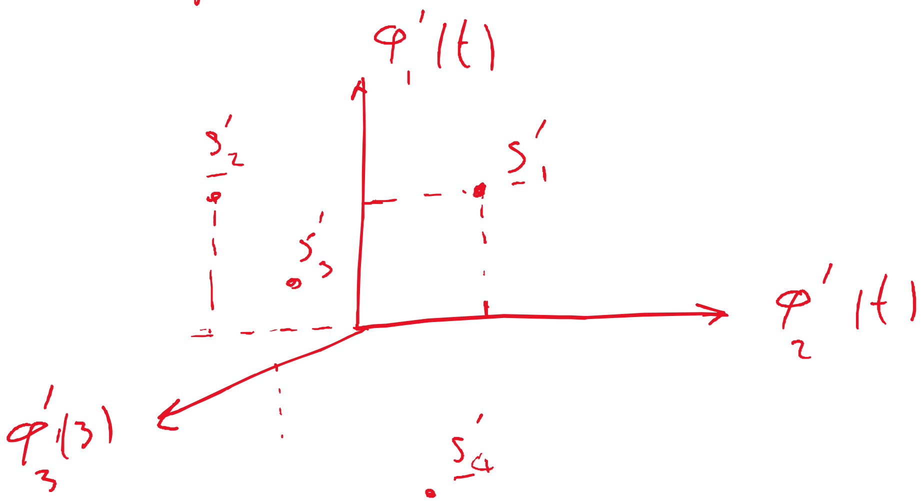
As we know, the Gram-schemith algorithm result is not unique. By inspection, we can find another result for this problem. $\{s_m(t)\}_{m=1}^{4}$ can represent by the linear combination of below complete orthonormal set $\{\varphi_i'(t)\}_{i=1}^{3}$



$$
\left.
\begin{aligned}
s_1(t) &= \varphi'_1(t) + \varphi'_2(t) &&\longrightarrow \quad \underline{s}'_1 = (1, 1, 0)^T \\
s_2(t) &= \varphi'_1(t) - \varphi'_2(t) &&\longrightarrow \quad \underline{s}'_2 = (1, -1, 0)^T \\
s_3(t) &= \varphi'_1(t) + \varphi'_2(t) - \varphi'_3(t) &&\longrightarrow \quad \underline{s}'_3 = (1, 1, -1)^T \\
s_4(t) &= -\varphi'_1(t) - \varphi'_2(t) - \varphi'_3(t) &&\longrightarrow \quad \underline{s}'_4 = (-1, -1, -1)^T
\end{aligned}
\right\}
$$

another signal constellation that is a rotation of previous constellation

Sketch this Constellation as an assignment and Compare it to the previous sketch.



3-dimentional

Check the vector lengths and their inner products